INFINITE SERIES INTRODUCTION TO SERIES

MATHEMATICS

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CONTENTS

Learning Outcomes

- Able to get clear idea about the real numbers and real valued functions.

- Obtain the skills of analyzing the concepts and applying appropriate methods for testing convergence of a series.

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Infinite Series: - Let $\{u_n\}$ be a sequence of real numbers.

Let $s_1=u_1$

$s_2=u_1+u_2$

$s_3=u_1+u_2+u_3$

……………

$s_n=u_1+u_2+u_3+\text{------}+u_n$

…………..

then the sequence $\{s_n\}$ is called an infinite series. The number u_n is called the n^{th} term of the series. The number s_n is called the n^{th} partial sum of the series. The infinite series $\{s_n\}$ is denoted by

$$\sum_{n=1}^{\infty} u_n \quad \text{(or)} \quad u_1+u_2+u_3+\ldots\ldots\ldots \quad \text{(or)} \quad \sum u_n .$$

Some times the infinite series begins with u_0. In this case

$s_1=u_0+u_1$

$s_2= u_0+u_1+u_2$

$s_3= u_0+u_1+u_2+u_3$

…………

$s_n= u_0+u_1+u_2+u_3+\text{------}+u_n$

…………

In this case we write the series as $\sum_{n=0}^{\infty} u_n$ If the sequence $\{s_n\}$ converges to l then we say that $\sum u_n$ converges to l. The number l is called the sum of the series and we write $\sum_{n=0}^{\infty} u_n = l$.

If the sequence $\{s_n\}$ diverges then we say that $\sum u_n$ diverges.

Results

1. If $\sum u_n$ converges to A, $\sum v_n$ converges to B then $\sum (u_n + v_n)$ converges to A+B
2. If $\sum u_n$ converges to A, $k \in R$ then $\sum k u_n$ converges to kA
3. If $\sum u_n$ diverges and $k \in R$, $k \neq 0$ then $\sum k u_n$ diverges to ∞

4. If $\sum u_n$ and $\sum v_n$ diverges then $\sum (u_n + v_n)$ diverges

5. If $\sum u_n$ converges and $\sum v_n$ diverges then $\sum (u_n + v_n)$ diverges

6. If $\sum u_n$ converges then $\lim\limits_{n \to \infty} u_n = 0$

The converse of the above result need not be true i.e., if

$$\lim_{n \to \infty} u_n = 0 \text{ then } \sum u_n \text{ may or may not converge.}$$

Note: If $\lim\limits_{n \to \infty} u_n \neq 0$ then $\sum u_n$ is divergent

Problems

1. Prove that $\sum \frac{n+1}{n+2}$ is divergent.

A. Here $u_n = \frac{n+1}{n+2}$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n+1}{n+2} = \lim_{n \to \infty} \frac{1+1/n}{1+2/n} = 1 \neq 0$$

$\therefore \sum u_n$ is divergent.

2. Prove that $\sum \frac{n}{2n+1}$ is divergent.

A. Here $u_n = \frac{n}{2n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$$

$\therefore \sum u_n$ is divergent

Series of non negative terms:-

If $\sum u_n$ is a series of non negative terms then $u_n \geq 0, \forall n$

If $\sum u_n$ is a series of positive terms then $u_n > 0, \forall n$

1) Geometric series:- The series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + --------$ is called geometric series.

This series converges if $0 < r < 1$ and diverges if $r \geq 1$.

2) Auxiliary series (or) p-series:- The series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+--------$ ($p\in R$) is called auxiliary series.

This series is convergent if $p>1$ and divergent if $p\leq 1$

Eg:- (1) $\sum_{n=0}^{\infty}\frac{1}{2^n}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent. Here $r=\frac{1}{2}$ and $0<\frac{1}{2}<1$

(2) $\sum_{n=0}^{\infty}3^n$ is divergent. Here $r=3$ and $3>1$

(3) $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. Here $p=2>1$

(4) $\sum_{n=1}^{\infty}n^2$ is divergent. Here $p=-2<1$

(5) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=0}^{\infty}\frac{1}{n^{1/2}}$ is divergent. Here $p=\frac{1}{2}<1$

(6) $\sum\frac{1}{n}$ is divergent. Here $p=1$

$\sum_{n=0}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ .... is called harmonic series.

(7) $\sum_{n=1}^{\infty} 1 = \sum \frac{1}{n^0}$ is divergent. Here p=0

(8) $\sum_{n=1}^{\infty} \frac{1}{n.n^{\frac{1}{100}}} = \sum_{n=0}^{\infty} \frac{1}{n^{\frac{101}{100}}}$ is convergent. Here p>1

Comparison test (1st type):-

Theorem (1):- If $\sum u_n$ and $\sum v_n$ are two series of positive terms such that

(i) $\exists$ a +ve integer m and $k \in R^+$ such that $u_n \leq kv_n, \forall n \geq m$ (ii)

$\sum v_n$ is convergent then $\sum u_n$ is also convergent.

Theorem (2):- If $\sum u_n$ and $\sum v_n$ are two series

(i) $\exists$ a +ve integer m and $k \in R^+$ such that $u_n \geq kv_n, \forall n \geq m$ (ii)

$\sum v_n$ is divergent then $\sum u_n$ is also divergent.

Problems

1. Test the convergence of $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$

A. Let $u_n = \frac{1}{n^2+1}$

We know that $n^2+1>n^2$, $\forall$ $n\geq1$

$$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$$

$$\Rightarrow u_n < v_n \forall n \geq 1 \text{ where } v_n = \frac{1}{n^2}$$

Now $\sum v_n = \sum \frac{1}{n^2}$ is convergent

$\therefore$ By comparison test first type, $\sum u_n$ is convergent.

2. Test the convergence of $\sum \frac{1}{2n^3-1}$

A. Let $u_n = \frac{1}{2n^3-1}$

We know that $2n^3 > n^3$, $\forall n \geq 1$

$$\Rightarrow \quad 2n^3 - 1 \geq n^3$$

$$\Rightarrow \frac{1}{2n^3 - 1} \leq \frac{1}{n^3}$$

$$\Rightarrow \quad u_n < v_n \forall n \geq 1 \text{ where } v_n = \frac{1}{n^3}$$

Now $\sum v_n = \sum \frac{1}{n^3}$ is convergent

$\therefore$ By comparison test first type, $\sum u_n$ is convergent.

END